LOW-DIMENSIONAL SURGERY AND THE YAMABE INVARIANT

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Abstract. Assume that M is a compact n-dimensional manifold and that N is obtained by surgery along a k-dimensional sphere, $k \leq n-3$. The smooth Yamabe invariants $\sigma(M)$ and $\sigma(N)$ satisfy $\sigma(N) \geq \min(\sigma(M), \Lambda)$ for a constant $\Lambda > 0$ depending only on n and k. We derive explicit lower bounds for Λ in dimensions where previous methods failed, namely for $(n,k) \in \{(4,1),(5,1),(5,2),(6,3),(9,1),(10,1)\}$. With methods from surgery theory and bordism theory several gap phenomena for smooth Yamabe invariants can be deduced.

Contents

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1. Introduction and Results

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Its scalar curvature will be denoted by s^g. We define the Yamabe functional by

$$\mathcal{F}^g(u) := \frac{\int_M \left(a_n |du|_g^2 + s^g u^2\right) dv^g}{\left(\int_M |u|^{p_n} dv^g\right)^{\frac{2}{p_n}}},$$

where $u \in C_c^\infty(M)$ does not vanish identically, and where $a_n := \frac{4(n-1)}{n-2}$ and $p_n := \frac{2n}{n-2}$. The *conformal Yamabe constant* $\mu(M, g)$ of (M, g) is then defined by

$$\mu(M, g) := \inf_{u \in C_c^\infty(M), u \not\equiv 0} \mathcal{F}^g(u).$$

This functional played a crucial role in the solution of the Yamabe problem which consists in finding a metric of constant scalar curvature in a given conformal class. For a compact manifold M the *Yamabe invariant* is defined by

$$\sigma(M) := \sup \mu(M, g),$$

where the supremum runs over all the metrics on M, or equivalently over all conformal classes on M. In order to stress that the Yamabe invariant only depends on the differentiable structure of M, it is often called the "smooth Yamabe invariant of M". One motivation for studying such an invariant is given by the following well-known result

Proposition 1.1. *A compact differentiable manifold of dimension $n \geq 3$ admits a metric with positive scalar curvature if and only if $\sigma(M) > 0$.*

Note that all manifolds in this article are manifolds *without boundary*.
We recall that classification of all compact manifolds of dimension $n \geq 3$ admitting a positive scalar curvature metric is a challenging open problem solved only in dimension 3 by using Hamilton's Ricci flow and Perelman's methods. This is one reason why much work has been devoted to the study of $\sigma(M)$.
One of the first goals should be to compute $\sigma(M)$ explicitly for some standard manifolds M. This is unfortunately a problem out of range even for what could be considered the simplest examples. For example, the value of the Yamabe invariant is not known for quotients of spheres except for $\mathbb{R}P^3$ (and the spheres themselves), for products of spheres of dimension at least 2 and for hyperbolic spaces of dimension at least 4.
One also could ask for general bounds for $\sigma(M)$. The fundamental one is due to Aubin,

$$\sigma(M) \leq \sigma(S^n) = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}.$$

Here $\mathbb{S}^n$ is the standard sphere in $\mathbb{R}^{n+1}$, and its volume is denoted by ω_n.
Unfortunately, in dimension $n \geq 5$, not much more is known. Even the basic question whether there exists a compact manifold M of dimension $n \geq 5$ satisfying $\sigma(M) \neq 0$ and $\sigma(M) \neq \sigma(S^n)$ is still open. It would also be interesting to see whether the set

$$\mathcal{S}_n(0) := \{\sigma(M) \mid M \text{ is a compact connected manifold of dimension } n\}$$

is finite or countably infinite, and whether $\mathcal{S}_n(0)$ is dense in $(-\infty, \sigma(S^n)]$. Much more is known now about

$$\mathcal{S}_n(i) := \{\sigma(M) \mid M \text{ is a compact } i\text{-connected manifold of dimension } n\}$$

for $i \geq 1$, as we will see below.
A useful tool for understanding the Yamabe invariant is to study its change under surgery type modifications of the manifold. The main results obtained this way are the following.

- In 1979, Gromov-Lawson and Schoen-Yau independently proved that the positivity of $\sigma(M)$ is preserved under surgery of dimension $k \leq n-3$. One important corollary is that any compact simply connected non-spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric. Together with results by Stephan Stolz (1992) this implies $\mathcal{S}_n(1) \subset (0, \sigma(S^n)]$ for $n \equiv 3, 5, 6, 7$ modulo 8, $n \geq 5$.
- In 1987, Kobayashi proved that 0-dimensional surgeries do not decrease $\sigma(M)$.
- In 2000, Petean and Yun proved that if N is obtained by a k-dimensional surgery ($k \leq n-3$) from M then $\sigma(N) \geq \min(0, \sigma(M))$. This implies in particular that if M is simply connected and has dimension $n \geq 5$ then $\sigma(M) \geq 0$. In other words $\mathcal{S}_n(1) \subset [0, \sigma(S^n)]$ for all $n \geq 5$.

In [4] we proved a generalization of these three results.

Theorem 1.2 ([4], Corollary 1.4)**.** *If N is obtained from a compact n-dimensional manifold M by a k-dimensional surgery, $k \leq n-3$, then*

$$\sigma(N) \geq \min(\Lambda_{n,k}, \sigma(M))$$

where $\Lambda_{n,k} > 0$ depends only on n and k. In addition, $\Lambda_{n,0} = \sigma(S^n)$.

As a corollary we see that 0 is not an accumulation point of $\mathcal{S}_n(1)$, $n \geq 5$, in other words we find that for any simply connected compact manifold M of dimension $n \geq 5$

- $\sigma(M) = 0$ if M is spin and if its index in KO_n does not vanish,
- $\sigma(M) \geq \alpha_n$, otherwise, where $\alpha_n > 0$ depends only on n.

Many other consequences can be deduced, see [4, Section 1.4], but one could find these results unsatisfactory, since the constant $\Lambda_{n,k}$ were not computed in [4] unless for $k = 0$. This effect was then reflected in the applications. For example, no explicit positive lower bound for the constant α_n above was known. The results in [3] and [2] yield explicit positive lower bounds for $\Lambda_{n,k}$ in the cases $2 \leq k \leq n-4$. In order to apply standard surgery techniques, it would be helpful to have lower bounds in the cases $k = 1$ and $k = n-3$.
The method established in the present article yields explicit positive lower bounds for all cases $k = 1 \leq n-4$ and in the cases $(n,k) = (6,3)$, $(n,k) = (5,2)$ and $(n,k) = (4,1)$. However it requires as input data a lower bound on the conformal Yamabe constant $\mu(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$. Such input data is provided in [17] and [18] in the cases $(n,k) \in \{(4,1), (5,1), (5,2), (9,1), (10,1)\}$. Unfortunately their method has to be strongly modified for each pair of dimensions, and as a courtesy to us, Petean and Ruiz provided the above cases, as these are the ones which will lead to interesting applications in Section 5.
We obtain in Corollary 5.3 that $\mathcal{S}_5(1) \subset (45.1, \sigma(S^5)]$, in other words: any compact simply connected manifold of dimension 5 satisfies

$$45.1 < \sigma(M) \leq \mu(\mathbb{S}^5) < 79.$$

In the same way, Corollary 5.4 says that $\mathcal{S}_6(1) \subset (49.9, \sigma(S^6)]$.

In dimensions $n \geq 7$ an unsolved problem persists for surgeries of codimension 3, i.e. for $n = k - 3$, see [2] for details about this problem.
This problem can be avoided by restricting to 2-connected manifolds. Together with results from [2] we obtain an explicit positive number t_n such that any compact n-dimensional 2-connected manifold M with vanishing index, $n \neq 4$, satisfies $\sigma(M) \geq t_n$, see Table 2 and Proposition 5.7. We thus see $\mathcal{S}_n(2) \subset \{0\} \cup [t_n, \sigma(S^n)]$ for all $n \neq 4$.

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2. Preliminaries

2.1. **Notation and model spaces.** We denote the standard flat metric on $\mathbb{R}^v$ by ξ^v. On the sphere $S^w \subset \mathbb{R}^{w+1}$ the standard round metric is denoted by ρ^w. The volume of (S^w, ρ^w) is

$$\omega_w = \frac{2\pi^{(w+1)/2}}{\Gamma\left(\frac{w+1}{2}\right)}.$$

Let $\mathbb{H}_c^v$ be the v-dimensional complete 1-connected Riemannian manifold with sectional curvature $-c^2$. The Riemannian metric on $\mathbb{H}_c^v$ is denoted by η_c^v. We fix a point x_0 in $\mathbb{H}_c^v$.
Next, we define the model spaces $\mathbb{M}_c$ through $\mathbb{M}_c := \mathbb{H}_c^v \times \mathbb{S}^w$, which has the Riemannian metric $G_c := \eta_c^v + \rho^w$. Note that in our previous articles [4, 3] we used the notation $\mathbb{M}_c^{v+w,v-1}$ for $\mathbb{M}_c$. Set $n := v + w$.
Let (N, h) be a Riemannian manifold of dimension n. Let Δ^h denote the non-negative Laplacian on (N, h). For $i = 1, 2$ we let $\Omega^{(i)}(N, h)$ be the set of non-negative C^2 functions u solving the Yamabe equation

$$a_n \Delta^h u + s^h u = \mu u^{p_n - 1} \tag{1}$$

for some $\mu = \mu(u) \in \mathbb{R}$ and satisfying

- $u \not\equiv 0$,
- $\|u\|_{L^{p_n}(N)} \leq 1$,
- $u \in L^\infty(N)$,

and

- $u \in L^2(N)$, for $i = 1$,

or

- $\mu(u)\|u\|_{L^\infty(N)}^{p_n-2} \geq \frac{(n-k-2)^2(n-1)}{8(n-2)}$, for $i = 2$.

For $i = 1, 2$ we set

$$\mu^{(i)}(N, h) := \inf_{u \in \Omega^{(i)}(N,h)} \mu(u).$$

In particular, if $\Omega^{(i)}(N, h)$ is empty then $\mu^{(i)}(N, h) = \infty$.
Finally, the constants in the surgery theorem are defined as follows. For integers $n \geq 3$ and $0 \leq k \leq n - 3$ set

$$\Lambda_{n,k}^{(i)} := \inf_{c \in [0,1]} \mu^{(i)}(\mathbb{M}_c)$$

and

$$\Lambda_{n,k} := \min\left\{\Lambda^{(1)}_{n,k}, \Lambda^{(2)}_{n,k}\right\}.$$

where $v = k+1$ and $w = n-k-1$.

2.2. **Strategy of proof.** The strategy we have used to find lower bounds of $\Lambda_{n,k}$ is the following.

- First prove that $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$. This was the main result in [2] which holds in the cases $k \leq n-4$ and $n = k+3 \in \{4,5\}$. For $n=6$, $k=3$, the results in [2] do not apply directly and just allow to prove that
$$\inf_{c\in[0,1)} \mu^{(2)}(\mathbb{M}_c) \geq \Lambda^{(1)}_{6,3}.$$
The case $c=1$ is treated separately: we exploit the fact that $\mathbb{M}_1$ is conformally equivalent to the standard sphere $\mathbb{S}^6 \setminus \mathbb{S}^3$ with a totally geodesic 3-sphere removed to show that $\mu^{(2)}(\mathbb{M}_1) \geq \mu(\mathbb{S}^6) \geq \Lambda^{(1)}_{6,3}$. We obtain again that $\Lambda^{(2)}_{6,3} \geq \Lambda^{(1)}_{6,3}$ (see Appendix B). It remains open whether the same holds for $n = k+3 \geq 7$.
- Find lower bounds for $\Lambda^{(1)}_{n,k}$. For this purpose, we show that $\mu^{(1)}(\mathbb{M}_c)$ can be estimated by the conformal Yamabe constant of the non-compact manifold $\mathbb{M}_c$, see Section 2.3. We are reduced to find a lower bound for conformal Yamabe constant of the product manifold $\mathbb{M}_c$. As mentioned before, there exists results in this direction; our paper [3] gives such a bound if $v \geq 3$ and $w \geq 3$. Also, the work of Petean and Ruiz apply if $w=1$. In this paper, we develop a method which completes the remaining cases.

The technical aspects of the argument in the present paper involve symmetrization and stretching maps to relate the the conformal Yamabe constants of $\mathbb{M}_c$ for different values of c. This is done in Section 3.

Remark 2.1. Our methods also apply to find explicit lower bounds for the conformal Yamabe constant of $\mathbb{H}^v_c \times (W,h)$, where (W,h) is any compact Riemannian manifold, i.e. if we replace the round sphere by (W,h). The case $(W,h) = \mathbb{S}^w$ is the only case for which we see applications, so for simplicity of presentation we restricted to this case.

2.3. **The generalized Yamabe functional of the model spaces.** For $u \in C^\infty(\mathbb{M}_c)$, $u \not\equiv 0$, we define the generalized Yamabe functional

$$\mathcal{F}^b_c(u) := \frac{\int_{\mathbb{M}_c}\left(a_n|du|^2 + bu^2\right)\, dv}{\|u\|^2_{L^{p_n}(\mathbb{M}_c)}}.$$

Clearly $\mathcal{F}^b_c(u) \geq \mathcal{F}^{b'}_c(u)$ if $b \geq b'$ and $\mathcal{F}^b_c(u) \geq \frac{b}{b'}\mathcal{F}^{b'}_c(u)$ if $0 < b \leq b'$.
The scalar curvature of $\mathbb{M}_c$ is $s_c := s^{G_c} = w(w-1) - c^2 v(v-1)$. The conformal Yamabe constant μ_c of $\mathbb{M}_c$ satisfies

$$\mu_c := \mu(\mathbb{M}_c) = \inf \mathcal{F}^{s_c}_c(u),$$

where the infimum is taken over all smooth functions u of compact support which do not vanish identically.
If u is a solution of (1) as in the definition of $\Omega^{(1)}(\mathbb{M}_c)$, then u is L^2 by assumption and thus also in the Sobolev space $H^{1,2}$. An integration by parts $\int u\Delta u\, dv =$

$\int |du|^2\, dv$ may then be performed in the integral defining $\mathcal{F}_c^b(u)$, and we conclude that

$$\mu^{(1)}(\mathbb{M}_c) \geq \mu_c.$$

Using $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$ and the definition of $\Lambda^{(1)}_{n,k}$ this implies positive lower bounds for $\Lambda_{n,k}$ for certain pairs (n,k), see Table 1.

2.4. **Symmetrization.** The group $\mathrm{Stab}_{x_0}(\mathrm{Isom}(\mathbb{H}^v_c))$ of isometries of $\mathbb{H}^v_c$ fixing $x_0 \in \mathbb{H}^v_c$ is isomorphic to $O(v)$ and we will fix such an isomorphism to identify $\mathrm{Stab}_{x_0}(\mathrm{Isom}(\mathbb{H}^v_c))$ with $O(v)$. A function on $\mathbb{H}^v_c$ is $O(v)$-invariant if and only if it depends only on the distance $d(\cdot, x_0)$ to the point x_0. A function on $\mathbb{M}_c$ is $O(v)$-invariant if and only if it depends only on $d(\cdot, x_0)$ and the point in $\mathbb{S}^w$.

Lemma 2.2. *For each $c \in [0,1]$*

$$\mu_c = \inf \mathcal{F}_c^{s_c}(\tilde{f})$$

where the infimum is taken over all $O(v)$-invariant functions on $\mathbb{M}_c$ which do not vanish identically.

Proof. The proof uses standard arguments and we just give a sketch. We must show that for any non-negative compactly supported smooth function $u : \mathbb{M}_c \to \mathbb{R}$ there is a $O(v)$-invariant non-negative compactly supported smooth function $\tilde{u} : \mathbb{M}_c \to \mathbb{R}$ satisfying $\mathcal{F}_c^{s_c}(\tilde{u}) \leq \mathcal{F}_c^{s_c}(u)$. If φ is a non-negative function on $\mathbb{H}^v_c$, there is a non-negative $O(v)$-invariant function φ^* defined on the same space called the *hyperbolic rearrangement of* φ, see [7]. This has the properties that for $p \geq 1$

$$\|\varphi^*\|_{L^p(\mathbb{H}^v_c)} = \|\varphi\|_{L^p(\mathbb{H}^v_c)}, \tag{2a}$$

$$\|\varphi_1^* - \varphi_2^*\|_{L^p(\mathbb{H}^v_c)} \leq \|\varphi_1 - \varphi_2\|_{L^p(\mathbb{H}^v_c)}, \tag{2b}$$

$$\|d\varphi^*\|_{L^p(\mathbb{H}^v_c)} \leq \|d\varphi\|_{L^p(\mathbb{H}^v_c)}, \tag{2c}$$

see [7, Section 4, Corollaries 1 and 3].
Let u be a non-negative function on $\mathbb{M}_c$. We set $\tilde{u}(\cdot, y) := (u(\cdot, y))^*$. From (2a) and (2c) we have $\|\tilde{u}\|_{L^{p_n}(\mathbb{M}_c)} = \|u\|_{L^{p_n}(\mathbb{M}_c)}$ and $\|d_{\mathbb{H}^v_c}\tilde{u}\|_{L^2(\mathbb{M}_c)} \leq \|d_{\mathbb{H}^v_c} u\|_{L^2(\mathbb{M}_c)}$. Let $\gamma : (-\varepsilon, \varepsilon) \to S^w$ be a curve. We apply (2b) with $\varphi_1 = u(\cdot, \gamma(t))$, $\varphi_2 = u(\cdot, \gamma(0))$, divide by $|t|$, and let t tend to 0. From this we conclude

$$\|d_{S^w}\tilde{u}(\gamma'(0))\|_{L^2(\mathbb{H}^v_c \times \{\gamma(0)\})} \leq \|d_{S^w} u(\gamma'(0))\|_{L^2(\mathbb{H}^v_c \times \{\gamma(0)\})}$$

and $\|d_{S^w}\tilde{u}\|_{L^2(\mathbb{M}_c)} \leq \|d_{S^w} u\|_{L^2(\mathbb{M}_c)}$. It follows that $\mathcal{F}_c^{s_c}(\tilde{u}) \leq \mathcal{F}_c^{s_c}(u)$ which ends the proof of Lemma 2.2. □

3. Comparing $\mathcal{F}_c^b$ to $\mathcal{F}_{c'}^{b'}$

We want to estimate $\mathcal{F}_c^b$ from below in terms of $\mathcal{F}_0^b$ and $\mathcal{F}_1^{b_1}$ for b_1 as large as possible.

3.1. **Comparing $\mathcal{F}_c^b$ to $\mathcal{F}_0^b$.** For $c \neq 0$ define $\mathrm{sh}_c(t) := c^{-1}\sinh(ct)$. In polar coordinates we have

$$\mathbb{H}^v_0 = \mathbb{R}^v = ((0,\infty) \times S^{v-1}, dt^2 + t^2\rho^{v-1}),$$

and

$$\mathbb{H}^v_c = ((0,\infty) \times S^{v-1}, dt^2 + \mathrm{sh}_c(t)^2\rho^{v-1}).$$

Lemma 3.1. *For $c > 0$ there is a unique diffeomorphism $f_c : [0,\infty) \to [0,\infty)$ for which the map $F_c : \mathbb{R}^v \to \mathbb{H}^v_c$ defined in polar coordinates as*

$$F_c : (t,\theta) \mapsto (f_c(t),\theta).$$

is volume preserving. Further $f_c'(t) \leq 1$ for all $0 \leq t < \infty$.

The map F_c squeezes the radial coordinate, so we will call F_c the *radial squeezing map from $\mathbb{R}^v$ to $\mathbb{H}^v_c$*.

Proof. The function

$$\varphi_c(r) := \left(\frac{v}{\omega_{v-1}}\mathrm{vol}\left(B^{\mathbb{H}^v_c}_{x_0}(r)\right)\right)^{1/v} = \left(v\int_0^r \mathrm{sh}_c(t)^{v-1}\,dt\right)^{1/v}$$

is a smooth function $[0,\infty) \to [0,\infty)$. Since $\varphi_c'(0) = 1$, $\varphi_c'(r) > 0$ for $r \geq 0$, and $\lim_{r\to\infty}\varphi_c(r) = \mathrm{vol}(\mathbb{H}^v_c) = \infty$ it is a diffeomorphism. We set $f_c := \varphi_c^{-1}$. Let $B_0(r)$ be the ball of radius r around 0 in $\mathbb{R}^v$. Since F_c is assumed to be volume preserving we have

$$\mathrm{vol}^{\mathbb{R}^v}(B_0(r)) = \mathrm{vol}^{\mathbb{H}^v_c}(F_c(B_0(r))),$$

or

$$\frac{\omega_{v-1}}{v}r^v = \omega_{v-1}\int_0^{f_c(r)} \mathrm{sh}_c(t)^{v-1}\,dt. \tag{3}$$

Differentiating (3) we get

$$r^{v-1} = f_c'(r)\,\mathrm{sh}_c(f_c(r))^{v-1}.$$

From (3) together with $\mathrm{sh}_c'(t) = \cosh(ct) \geq 1$ we find

$$\begin{aligned} r^v &= \int_0^{f_c(r)} v\,\mathrm{sh}_c(t)^{v-1}\,dt \\ &= \int_0^{f_c(r)} (\mathrm{sh}_c(t)^v)'\frac{1}{\mathrm{sh}_c'(t)}\,dt \\ &\leq \int_0^{f_c(r)} (\mathrm{sh}_c(t)^v)'\,dt \\ &= \mathrm{sh}_c(f(r))^v, \end{aligned}$$

so $r \leq \mathrm{sh}_c(f_c(r))$ and we conclude that $f_c'(r) \leq 1$. □

We extend the radial squeezing map to a volume preserving map $\widehat{F}_c : \mathbb{M}_0 \to \mathbb{M}_c$ by setting

$$\widehat{F}_c := F_c \times \mathrm{Id}_{\mathbb{S}^w} : \mathbb{R}^v \times \mathbb{S}^w \to \mathbb{H}^v_c \times \mathbb{S}^w.$$

Proposition 3.2. *For $O(v)$-invariant functions $u : \mathbb{M}_c \to \mathbb{R}$ we have*

$$\mathcal{F}^b_c(u) \geq \mathcal{F}^b_0(u \circ \widehat{F}_c).$$

Proof. The differential $d(u\circ\widehat{F}_c)$ decomposes orthogonally in a $\mathbb{R}^v$-component $d_{\mathbb{R}^v}(u\circ\widehat{F}_c)$ and a $\mathbb{S}^w$-component $d_{\mathbb{S}^w}(u\circ\widehat{F}_c)$. Similarly, du splits orthogonally in a $\mathbb{H}^v_c$-component $d_{\mathbb{H}^v_c}u$ and a $\mathbb{S}^w$-component $d_{\mathbb{S}^w}u$. Then $d_{\mathbb{R}^v}(u\circ\widehat{F}_c) = d_{\mathbb{H}^v_c}u \circ d\widehat{F}_c$ and $d_{\mathbb{S}^w}(u\circ\widehat{F}_c) = d_{\mathbb{S}^w}u \circ d\widehat{F}_c = d_{\mathbb{S}^w}u$. Thus

$$|d_{\mathbb{R}^v}(u\circ\widehat{F}_c)| = |d_{\mathbb{H}^v_c}u \circ d\widehat{F}_c| = |d_{\mathbb{H}^v_c}u|f'(t) \leq |d_{\mathbb{H}^v_c}u|$$

and

$$|d_{\mathbb{S}^w}(u \circ \widehat{F}_c)| = |d_{\mathbb{S}^w} u|.$$

It follows that $|d(u \circ \widehat{F}_c)| \leq |du|$. Further the volume form is preserved by the map $\widehat{F}_c$ and the Proposition follows. □

Corollary 3.3. *If $s_c > 0$ then $\mu_c \geq \frac{s_c}{s_0}\mu_0$.*

This corollary gives good estimates if c is sufficiently small, as then $s_c > 0$. However in case $v > w$ the corollary can no longer be applied for c close to 1.

3.2. **Comparing $\mathcal{F}_c^b$ to $\mathcal{F}_1^{b_1}$.** For $c > 0$ we define a diffeomorphism $R_c : \mathbb{H}_c^v \to \mathbb{H}_1^v$ by $R_c(t, \theta) = (ct, \theta)$. The map R_c is a c-homothety in the sense that the Riemannian metric of $\mathbb{H}_c^v$ is $\eta_c^v = c^{-2} R_c^* \eta_1^v$ where η_1^v is the Riemannian metric of $\mathbb{H}_1^v$. Taking the product with the identity map on the round sphere we obtain a map $\widehat{R}_c : \mathbb{M}_c \to \mathbb{M}_1$. The metric G_c on $\mathbb{M}_c$ is then given by $G_c = \widehat{R}_c^*(c^{-2}\eta_1^v + \rho^w)$.
The following Proposition is an extension of [4, Lemma 3.7].

Proposition 3.4. *If $c \in (0,1)$, then $\mathcal{F}_c^{c^2 s_1}(u \circ \widehat{R}_c) \geq c^{2w/n} \mathcal{F}_1^{s_1}(u)$ for all functions $u \in C_c^\infty(\mathbb{M}_1)$.*

Proof. We have

$$\begin{aligned}
|d(u \circ \widehat{R}_c)|_{G_c}^2 &= |R_c^*(du)|_{G_c}^2 \\
&= |du|^2_{c^{-2}\eta_1^v + \rho^w} \\
&= c^2 |d_{\mathbb{H}_c^v} u|^2_{\eta_1^v} + |d_{\mathbb{S}^w} u|^2_{\rho^w} \\
&\geq c^2 \left(|d_{\mathbb{H}_c^v} u|^2_{\eta_1^v} + |d_{\mathbb{S}^w} u|^2_{\rho^w} \right) \\
&= c^2 |du|^2_{g_1}.
\end{aligned}$$

In addition, $dv^{G_c} = c^{-v} \widehat{R}_c^* dv^{g_1}$. From this we find that

$$\begin{aligned}
\mathcal{F}_c^{c^2 s_1}(u \circ \widehat{R}_c) &= \frac{\int_{\mathbb{M}_c} \left(a_n |d(u \circ \widehat{R}_c)|^2_{G_c} + c^2 s_1 (u \circ \widehat{R}_c)^2 \right) dv^{G_c}}{\left(\int_{\mathbb{R}^v \times S^w} (u \circ \widehat{R}_c)^{p_n} \, dv^{G_c} \right)^{\frac{2}{p_n}}} \\
&\geq \frac{\int_{\mathbb{M}_1} \left(a_n c^2 |du|^2_{g_1} + c^2 s_1 u^2 \right) c^{-v} \, dv^{g_1}}{\left(\int_{\mathbb{R}^v \times S^w} u^{p_n} c^{-v} \, dv^{g_1} \right)^{\frac{2}{p_n}}} \\
&= c^{2w/n} \mathcal{F}_1^{s_1}(u),
\end{aligned}$$

which is the statement of the Proposition. □

To apply the proposition, note that

$$s_c = w(w-1) - c^2 v(v-1) \geq c^2(w(w-1) - v(v-1)) = c^2 s_1.$$

This implies

$$\mathcal{F}_c^{s_c}(u \circ \widehat{R}_c) \geq \mathcal{F}_c^{c^2 s_1}(u).$$

By taking the infimum over all non-vanishing smooth functions $u \in C_c^\infty(\mathbb{M}_1)$ with compact support we obtain the following.

Corollary 3.5. *For $c \in (0,1)$ we have*

$$\mu_c \geq c^{2w/n} \mu_1.$$

This estimate gives uniform estimates fur μ_c if c is bounded away from 0. Because of $\mu_1 = \mu(\mathbb{S}^n)$ we obtain explicit bounds in any dimension. However these bounds tend to 0 as $c \to 0$.

4. Conclusions

4.1. **Interpolation of the previous inequalities.** We now improve the bounds obtained in Corollaries 3.3 and 3.5 by combining Propositions 3.2 and 3.4 in an interpolation argument.

Theorem 4.1. *For all $c \in (0,1)$ we have*

$$\mu_c \geq \left(\frac{\mu_0}{\mu_1} - \frac{c^2 v(v-1)}{(1-c^2)w(w-1) + c^2 v(v-1)} \left(\frac{\mu_0}{\mu_1} - c^{2w/n} \right) \right) \mu_1 \tag{4}$$

and

$$\mu_c \geq c^{2w/n} \mu_1. \tag{5}$$

As discussed in Appendix A, Inequality (4) is stronger than Inequality (5) for $c^{2w/n} < \mu_0/\mu_1$ and Inequality (5) is stronger for $c^{2w/n} > \mu_0/\mu_1$.

Proof. Inequality (5) is the statement of Corollary 3.5. Assume that $\lambda \geq 0$ and $\tau \geq 0$ satisfy

$$\lambda + \tau \leq 1, \tag{6}$$

$$\lambda c^2 s_1 + \tau s_0 \leq s_c. \tag{7}$$

Then we get

$$\begin{aligned} \mathcal{F}_c^{s_c}(u) &\geq \lambda \mathcal{F}_c^{c^2 s_1}(u \circ \widehat{R}_c^{-1}) + \tau \mathcal{F}_c^{s_0}(u \circ \widehat{F}_c) \\ &\geq \lambda c^{2w/n} \mathcal{F}_1^{s_1}(u \circ \widehat{R}_c^{-1}) + \tau \mathcal{F}_0^{s_0}(u \circ \widehat{F}_c) \\ &\geq \lambda c^{2w/n} \mu_1 + \tau \mu_0, \end{aligned} \tag{8}$$

where we used Proposition 3.4 for the second inequality. It follows that

$$\mu_c \geq \lambda c^{2w/n} \mu_1 + \tau \mu_0. \tag{9}$$

The lines described by $\lambda + \tau = 1$ and $\lambda c^2 s_1 + \tau s_0 = s_c$ intersect in (λ_0, τ_0) where

$$\lambda_0 = \frac{v(v-1)}{(c^{-2}-1)w(w-1) + v(v-1)} \in (0,1), \qquad \tau_0 = 1 - \lambda_0, \tag{10}$$

see Appendix A. Setting $\lambda := \lambda_0$ and $\tau := \tau_0$ in (9) yields Inequality (4). $\square$

The estimates obtained by the theorem rely on explicit lower bounds for μ_0. Such lower bounds can be found in the literature in the following cases.

- (i) $v = 1$, $w \geq 2$. Then $\mu_0 = \mu_1 = \mu_c = \mu(\mathbb{S}^n)$ for all $c \in (0,1)$. This case is trivial as $\mathbb{R} \times \mathbb{S}^w$ is conformal to a round sphere of dimension $n = w + 1$ with two points removed.
- (ii) $(v, w) \in \{(2,2), (2,3), (2,7), (2,8), (3,2)\}$. In these cases bounds have been derived in [17, 18] using isoperimetric profiles.
- (iii) $v \geq 3$ and $w \geq 3$. See [3] where an explicit lower bound of the Yamabe functional of $\mathbb{R}^v \times \mathbb{S}^w$ in terms of the Yamabe functionals of $\mathbb{R}^v$ and $\mathbb{S}^w$ is used.

(iv) $v \geq 4$ and $w = 2$. This case is not explicitly written in [3] but can be deduced from the main result of that paper. We just observe that this result implies that

$$\mu_0 \geq \frac{n a_n}{(3a_3)^{\frac{3}{n}}((n-3)a_{n-3})^{\frac{n-3}{n}}} \mu(\mathbb{R}^{n-3})^{\frac{n-3}{n}} \mu(\mathbb{R} \times \mathbb{S}^2)^{\frac{3}{n}}$$

where $a_k := \frac{4(k-1)}{k-2}$ for $k \geq 3$. Next, note that $\mu(\mathbb{R}^{n-3}) = \mu(\mathbb{S}^{n-3})$ and since $\mathbb{R} \times \mathbb{S}^2$ is conformally equivalent to $\mathbb{S}^3$ with two points removed we have $\mu(\mathbb{R} \times S^2) = \mu(\mathbb{S}^3)$. Hence, we get

$$\mu_0 \geq \frac{n a_n}{24^{\frac{3}{n}}((n-3)a_{n-3})^{\frac{n-3}{n}}} \mu(\mathbb{S}^{n-3})^{\frac{n-3}{n}} \mu(\mathbb{S}^3)^{\frac{3}{n}}.$$

In the case $(v, w) = (4, 2)$ this leads to

$$\mu_0 \geq 0.56885\mu_1 > 54.77. \tag{11}$$

A similar argument also yields lower bounds for μ_0 in the cases $v - 2 \geq w \geq 3$. These bounds on μ_0 are slightly stronger than the ones in (iii).

The estimate is optimal in Case (i). In this case nothing remains to be proven, and we will not discuss it further. In Cases (ii) and (iii) the bound is not likely to be optimal. Any improvement of the lower bound for μ_0 would improve the bounds obtained in Theorem 4.1. In [3] a lower bound on μ_c is derived which is uniform in c. Thus Theorem 4.1 does not currently yield improved estimates in Case (iii). However, if a better lower bound for μ_0 is available, it might be relevant as well, and will be also considered in the following. The most important applications thus come in Case (ii).

4.2. **Analytical Conclusions.** We now want to derive concrete bounds on $\Lambda_{v+w,v-1}$ for special values of v and w.

Corollary 4.2. *For all $c \in [0, 1]$ and all $v \geq 2$ and $w \geq 2$ we obtain*

$$\mu_c \geq \left(1 - \frac{v(v-1)}{\left(\sqrt{v(v-1)} + \sqrt{w(w-1)}\right)^2}\right)\mu_0. \tag{12}$$

Proof. Using (4) and the facts that $\mu_1 > \mu_0$ and $c^{2w/n} \geq c^2$ we deduce

$$\mu_c \geq \left(1 - \frac{(1-c^2)c^2 v(v-1)}{(1-c^2)w(w-1) + c^2 v(v-1)}\right)\mu_0 \tag{13}$$

for general values of v and w. The right hand side attains its minimum over $c \in [0, 1]$ for

$$c^2 = \frac{\sqrt{w(w-1)}}{\sqrt{v(v-1)} + \sqrt{w(w-1)}},$$

from which (12) follows. □

Example 4.3. $v = 2$, $w = 3$: In [18, Theorem 1.4] Petean and Ruiz have obtained $\mu(\mathbb{R}^2 \times \mathbb{S}^3) \geq 0.75\mu(\mathbb{S}^5)$, that is $\mu_0 \geq 0.75\mu_1$. Using (12) we obtain

$$\mu_c \geq \frac{\sqrt{3}}{2}\mu_0 \geq 0.649\mu_1 \geq 51.2$$

Thus $\Lambda_{5,1} \geq 51.2$.
Compare this value with $\mu(\mathbb{S}^5) = 78.996...$

(v,w)	(n,k)	μ_0/μ_1	Analytic	Numeric	$\mu_1 = \mu(\mathbb{S}^n)$
$(2,2)$	$(4,1)$	0.68	38.9	38.9	61.56
$(2,3)$	$(5,1)$	0.75	51.2	56.6	79.00
$(2,7)$	$(9,1)$	0.747	106.9	109.2	147.87
$(2,8)$	$(10,1)$	0.626	100.6	102.6	165.02
$(3,2)$	$(5,2)$	0.63	29.7	45.1	79.00
$(4,2)$	$(6,3)$	0.568	36.4	49.9	96.30

TABLE 1. Lower estimates for $\inf \mu_c = \Lambda_{n,k}$. The fourth column shows the analytic estimates from Corollary 4.2 and 4.6. The fifth column shows the numerical estimates from Subsection 4.3. The value for μ_1 is approximate, whereas the lower bounds are rounded down.

Example 4.4. $v = 2$, $w = 7$: In [18, Theorem 1.6] Petean and Ruiz have obtained $\mu(\mathbb{R}^2 \times \mathbb{S}^7) \geq 0.747\mu(\mathbb{S}^9)$, that is $\mu_0 \geq 0.747\mu_1$. Using (12) we obtain

$$\mu_c \geq \left(1 - \frac{2}{(\sqrt{2}+\sqrt{42})^2}\right)\mu_0 \geq 0.723\mu_1 \geq 106.9$$

Thus $\Lambda_{9,1} \geq 106.9$
Compare this value with $\mu(\mathbb{S}^9) = 147.87...$

Example 4.5. $v = 2$, $w = 8$: In [18, Theorem 1.6] Petean and Ruiz have obtained $\mu(\mathbb{R}^2 \times \mathbb{S}^8) \geq 0.626\mu(\mathbb{S}^{10})$, that is $\mu_0 \geq 0.626\mu_1$. Using (12) we obtain

$$\mu_c \geq \left(1 - \frac{2}{(\sqrt{2}+\sqrt{56})^2}\right)\mu_0 \geq 0.610\mu_1 \geq 100.69$$

Thus $\Lambda_{10,1} \geq 100.69$.
Compare this value with $\mu(\mathbb{S}^{10}) = 165.02...$

In the case $v = w$ we find better estimates for the right hand side of (4).

Corollary 4.6. *Assume $v = w \geq 2$ and $\mu_0/\mu_1 \geq \gamma > 0$. Then*

$$\inf_{c\in[0,1]} \mu_c \geq \left(\gamma - \frac{4}{27}\gamma^3\right)\mu_1$$

Proof. Using $v = w$ we obtain directly from (4):

$$\mu_c \geq \left(\left(c - \frac{\mu_0}{\mu_1}\right)\frac{1}{c^{-2}} + \frac{\mu_0}{\mu_1}\right)\mu_1 = \left(c^3 - c^2\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_1}\right)\mu_1 \geq \left(c^3 - c^2\gamma + \gamma\right)\mu_1$$

for any $\gamma \in (0, \mu_0/\mu_1]$. On the interval $[0,1]$ the right hand side attains its minimum in $c = \frac{2}{3}\gamma$. This yields the statement of the corollary. $\square$

Example 4.7. For $v = w = 2$ Petean and Ruiz [17, Theorem 1.2] have derived the bound $\gamma = 0.68$. This yields

$$\Lambda_{4,1} \geq 0.63\mu_1 \geq 38.9.$$

In the case $v > w$ one can use $c^{2w/n} \geq c$ which improves inequality(13) to

$$\mu_c \geq \left(1 - \frac{(1-c)c^2 v(v-1)}{(1-c^2)w(w-1) + c^2 v(v-1)}\right)\mu_0$$

which again yields better estimates for the right hand side of (4).

Obviously in the case $(v, w) = (4, 2)$ the determination of the value c for which μ_c is minimal, gives the equation $5c^3 + 3c = 2$ which has as only real solution

$$c = \frac{1}{5}\sqrt[3]{25 + 5\sqrt{30}} + \frac{1}{\sqrt[3]{25 + 5\sqrt{30}}} \approx 0.48108.$$

Example 4.8. For $(v, w) = (4, 2)$ we have derived the bound $\gamma = 0.56885$, see equation (11). This yields

$$\Lambda_{6,3} \geq 0.3788\mu_1 \geq 36.4$$

The explicit values deduced from the above corollaries are summarized in Table 1.

4.3. **Numerical Conclusions.** Numerical computations yield better bounds. Such improved bounds are important for applications, especially for some particular values, as for example the case $v = 3$, $w = 2$.

Using the procedure "Minimize" from the "Optimization" package of the program Maple 13.0 we numerically minimized the right hand side of (4). The results of this calculation provided the bounds given in the column "Numeric" of Table 1.

Example 4.9. Assume $v = 3$ and $w = 2$. In [18, Theorem 1.4] Petean and Ruiz have obtained $\mu(\mathbb{R}^3 \times \mathbb{S}^2) \geq 0.63\mu(\mathbb{S}^5)$, that is $\mu_0 \geq 0.63\mu_1$. A numerical evaluation of (4) yields

$$\inf_{c\in[-1,1]} \mu_c \geq 0.571\mu_1 > 45.1,$$

and we conclude that $\Lambda_{5,2} > 45.1$.

Example 4.10. Assume $v = 2$ and $w = 7$. In [18, Theorem 1.6] Petean and Ruiz have obtained $\mu(\mathbb{R}^2 \times \mathbb{S}^7) \geq 0.747\mu(\mathbb{S}^9)$, that is $\mu_0 \geq 0.747\mu_1$. A numerical evaluation of (4) yields

$$\inf_{c\in[-1,1]} \mu_c \geq 0.739\mu_1 > 109.2,$$

and we conclude that $\Lambda_{9,1} > 109.2$.

Example 4.11. Assume $v = 2$ and $w = 8$. In [18, Theorem 1.6] Petean and Ruiz have obtained $\mu(\mathbb{R}^2 \times \mathbb{S}^8) \geq 0.626\mu(\mathbb{S}^{10})$, that is $\mu_0 \geq 0.626\mu_1$. A numerical evaluation of (4) yields

$$\inf_{c\in[-1,1]} \mu_c \geq 0.622\mu_1 > 102.6$$

and we conclude that $\Lambda_{10,1} > 102.6$.

Example 4.12. Assume $v = 4$ and $w = 2$. In (11) we have seen that $\mu(\mathbb{R}^4 \times \mathbb{S}^2) \geq 0.56885\mu(\mathbb{S}^6)$, that is $\mu_0 \geq 0.56885\mu_1$. A numerical evaluation of (4) yields

$$\inf_{c\in[-1,1]} \mu_c \geq 0.51909\mu_1 > 49.98$$

and we conclude that $\Lambda_{10,1} > 102.6$.

Similar bounds for other dimensions could also be obtained using the same method. We will see that the cases derived as examples above have interesting topological applications.

4.4. **Bibliographic remark.** At the time when this article went into press, there was important progress connected to the Yamabe constant $\mu_c = \mu(\mathbb{M}_c)$: Solutions of the Yamabe equation on $\mathbb{M}_c$ which are constant on the sphere component, were studied systematically in [10].

5. Topological applications

The lower bounds for $\Lambda_{n,1}$, $n \in \{4, 5, 9, 10\}$, and $\Lambda_{5,2}$ and $\Lambda_{6,3}$ lead to estimates of the Yamabe invariant for certain classes of manifolds.

5.1. **Applications of the lower bound for $\Lambda_{5,2}$.** The following two propositions are standard consequences of the methods developed for the proof of the h-cobordism theorem. A proof for a similar statement can be found in [13, Theorem IV.4.4, pages 299–300]. As we do not know of a reference for the formulations given here we include their proofs.

Proposition 5.1. *Let M_0 and M_1 be non-empty, compact, connected, and simply connected spin manifolds of dimension $n \geq 5$. Assume that M_0 and M_1 are spin bordant. Then one can obtain M_1 from M_0 by a sequence of surgeries of dimensions ℓ where $2 \leq \ell \leq n-3$.*

Proof. Let W be a spin bordism from M_0 to M_1. By surgeries in the interior we simplify W to be connected, simply connected, and have $\pi_2(W) = 0$ (one then says W is 2-connected). Then $H_i(W, M_j) = 0$ for $i = 0, 1, 2$. We apply [12, VIII Thm. 4.1] for $k = 3$ and $m = n+1$. One obtains that there is a handle presentation of the bordism such that for any $i < 3$ and any $i > n-2$ the number of i-handles is given by $b_i(W, M_0)$. Any i-handle corresponds to a surgery of dimension $i-1$. It remains to show that $b_i(W, M_0) = 0$ for $i \in \{0, 1, 2, n+1, n, n-1\}$. This is trivial for $i \in \{0, 1, 2\}$. By Poincaré duality $H^{n+1-i}(W, M_0)$ is dual to $H_i(W, M_1)$ which vanishes for $i = 0, 1, 2$. On the other hand the universal coefficient theorem tells us that the free parts of $H^i(W, M_0)$ and $H_i(W, M_0)$ are isomorphic. Thus $b_i(W, M_0)$ which is by definition the rank of (the free part of) $H_i(W, M_0)$ vanishes for $i \in \{n+1, n, n-1\}$. □

Proposition 5.2. *Let M_0 and M_1 be non-empty compact connected and simply connected non-spin manifolds of dimension $n \geq 5$, and assume that these manifolds are oriented bordant. Then one can obtain M_1 from M_0 by a sequence of surgeries of dimensions ℓ, $2 \leq \ell \leq n-3$.*

Proof. The proof is similar to the proof in the spin case. However the bordism W cannot be simplified to $\pi_2(W) = 0$, but only to $\pi_2(W) = \mathbb{Z}/2\mathbb{Z}$ with surjective maps $\pi_2(M_j) \to \pi_2(W)$. This implies again that $H_i(W, M_j) = 0$ for $i = 0, 1, 2$, and $j = 1, 2$. The proof continues exactly as in the spin case. □

Corollary 5.3. *Let M be a compact simply connected manifold of dimension 5, then*

$$45.1 < \sigma(M) \leq \mu(\mathbb{S}^5) < 79.$$

Proof. The upper bound for $\sigma(M)$ is standard.
To prove the lower bound we consider first the case when M is spin. As the 5-dimensional spin bordism group Ω_5^{Spin} is trivial, M is the boundary of a compact 6-dimensional spin manifold. By removing a ball we obtain a spin bordism from S^5 to M. Using Proposition 5.1 we see that M can be obtained by 2-dimensional surgeries from S^5. As a consequence $\sigma(M) \geq \Lambda_{5,2} > 45.1$.
Next we consider the case when M is not spin. The oriented bordism group Ω_5^{SO} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ represents a non-trivial element in Ω_5^{SO}. Thus M is either oriented bordant to the empty set or to $\mathrm{SU}(3)/\mathrm{SO}(3)$.

We consider now the case that M is oriented bordant to SU(3)/SO(3). By Appendix C we see that $\sigma(\mathrm{SU}(3)/\mathrm{SO}(3)) > 64$. Since SU(3)/SO(3) is not spin Proposition 5.2 implies that we can obtain M from SU(3)/SO(3) by a finite number of 2-dimensional surgeries. Thus

$$\sigma(M) \geq \min\left(\Lambda_{5,2}, \sigma(\mathrm{SU}(3)/\mathrm{SO}(3))\right) > 45.1.$$

It remains to consider the case that M is oriented bordant to the empty set, or equivalently to S^5. However, S^5 is spin and cannot be used to apply Proposition 5.2. Instead we use the space SU(3)/SO(3)#SU(3)/SO(3) which is simply connected, non-spin and an oriented boundary. By [11, Theorem 2] we know that $\sigma(\mathrm{SU}(3)/\mathrm{SO}(3)\#\mathrm{SU}(3)/\mathrm{SO}(3)) \geq \sigma(\mathrm{SU}(3)/\mathrm{SO}(3))$. We apply Proposition 5.2 with $M_0 = \mathrm{SU}(3)/\mathrm{SO}(3)\#\mathrm{SU}(3)/\mathrm{SO}(3)$ and $M_1 = M$ and thus we obtain M from M_0 by a finite number of 2-dimensional surgeries. From this we find

$$\sigma(M) \geq \min\left(\Lambda_{5,2}, \sigma(\mathrm{SU}(3)/\mathrm{SO}(3))\right) > 45.1$$

which concludes the proof of the corollary. □

Let us compare the lower bound 45.1 for simply connected 5-manifolds to the expected values for the smooth Yamabe invariant on non-simply-connected spherical space forms in dimension 5. Assume that $M = S^5/\Gamma$ where the finite group $\Gamma \subset \mathrm{SO}(6)$ acts freely on S^5. It was conjectured by Schoen [19, Page 10, lines 6–11] that on such manifolds the supremum in the definition of the smooth Yamabe number is attained by the standard conformal structure. If this is true, then $\sigma(\mathbb{R}P^5)$ would be equal to $45.371\ldots$. Except S^5 and $\mathbb{R}P^5$ all 5-dimensional space forms would have σ-invariant below 45.1.

5.2. Applications of the lower bound for $\Lambda_{6,3}$.

Corollary 5.4. *Let M be a compact simply connected manifold of dimension 6, then*

$$49.9 < \sigma(M) \leq \mu(\mathbb{S}^6) < 96.30.$$

Proof. The proof of this corollary is a straightforward adaptation of the proof of previous corollary, using the fact that both the spin bordism group and the oriented bordism group are trivial in dimension 6. We obtain

$$\sigma(M) \geq \min(\Lambda_{6,2}, \Lambda_{6,3}) \geq 49.9.$$

□

5.3. Applications of the lower bound for $\Lambda_{9,1}$ and $\Lambda_{10,1}$ to spin manifolds.

For a compact spin manifold M of dimension n the alpha-genus $\alpha(M) \in KO_n$ is equal to the index of the Clifford-linear Dirac operator on M. It depends only on the spin bordism class of M.

Lemma 5.5. *Let M be a compact 2-connected spin manifold of dimension $n \in \{9, 10\}$ which has $\alpha(M) = 0$. Then M is obtained from S^9 or $\mathbb{H}P^2 \times S^1$ (for $n = 9$) or from S^{10} or $\mathbb{H}P^2 \times S^1 \times S^1$ (for $n = 10$) by a sequence of surgeries of dimensions $k \in \{0, 1, \ldots, n-4\}$. All these surgeries are compatible with orientation and spin structure.*

Note that S^1 carries two spin structure. One spin structure is obtained from the spin structure on D^2 by restriction to the boundary $S^1 = \partial D^2$, and it is called the *bounding spin structure*. In the above lemma we assume that all manifolds S^1 are equipped with the other spin structure, the *non-bounding spin structure*.

Proof. From the description of the Spin bordism group in [5] and [6] we know that M is spin bordant to $P = \emptyset$ or to $P = \mathbb{H}P^2 \times S^1$ (if $n = 9$) and M is spin bordant to $P = \emptyset$ or to $P = \mathbb{H}P^2 \times S^1 \times S^1$ (if $n = 10$).
Now let W be a spin bordism from P to M. By performing surgeries of dimension 0, 1, 2, and 3 one can find a spin bordism W' from P to M which is 3-connected, that is W' is connected and $\pi_1(W') = \pi_2(W') = \pi_3(W') = 0$. The inclusion $i : M \to W$ is thus 3-connected, that is bijective on π_i for $i \leq 2$ and surjective on π_3. This implies that W' can be decomposed into handles each of which corresponds to a surgery of dimension $\leq n - 4$. □

The following corollary extends similar results from [2] which hold in dimension $n = 7$, $n = 8$ and $n \geq 11$. We define $s_1 := \sigma(\mathbb{H}P^2 \times S^1)$ and $s_2 := \sigma(\mathbb{H}P^2 \times S^1 \times S^1)$.

Corollary 5.6. *Let M be a 2-connected compact spin manifold of dimension $n = 9$ or $n = 10$ with $\alpha(M) = 0$. Then*

$$\sigma(M) \geq \begin{cases} \min\{\Lambda_{9,1}, \Lambda_{9,2}, \Lambda_{9,3}, \Lambda_{9,4}, \Lambda_{9,5}, s_1\} > 109.2 & \text{for } n = 9, \\ \min\{\Lambda_{10,1}, \Lambda_{10,2}, \Lambda_{10,3}, \Lambda_{10,4}, \Lambda_{10,5}, \Lambda_{10,6}, s_2\} \geq 97.3 & \text{for } n = 10. \end{cases}$$

Proof. Lemma 5.5 implies

$$\sigma(M) \geq \min\{\Lambda_{9,1}, \Lambda_{9,2}, \Lambda_{9,3}, \Lambda_{9,4}, \Lambda_{9,5}, s_1\}$$

if $n = 9$ and

$$\sigma(M) \geq \min\{\Lambda_{10,1}, \Lambda_{10,2}, \Lambda_{10,3}, \Lambda_{10,4}, \Lambda_{10,5}, \Lambda_{10,6}, s_2\}$$

if $n = 10$. The relations $\Lambda_{9,1} > 109.2$ and $\Lambda_{10,1} > 102.6$ follow from Examples 4.10 and 4.11. The relations

$$\min\{\Lambda_{9,2}, \Lambda_{9,3}, \Lambda_{9,4}, \Lambda_{9,5}\} > 109.4 > 109.2$$

and

$$\min\{\Lambda_{10,2}, \Lambda_{10,3}, \Lambda_{10,4}, \Lambda_{10,5}, \Lambda_{10,6}\} > 126.4 > 102.6$$

follow from the product formula, see [3, Corollary 3.3]. From [1, Theorem 1.1] it follows that $s_k \geq \mu(\mathbb{H}P^2 \times \mathbb{R}^k)$. To estimate s_1 for $n = 9$ we apply results of [16]. The quantities V and V_8 in that paper satisfy

$$(\frac{V}{V_8})^{2/9} = 0.9370...,$$

see Appendix D. Thus, [16, Theorem 1.2] tells us that

$$s_1 \geq \mu(\mathbb{H}P^2 \times \mathbb{R}) \geq 0.9370\mu(\mathbb{S}^9) = 138.57... > 109.2.$$

An estimate for s_2 when $n = 10$ is provided by [18, Example after Theorem 1.7], namely

$$s_2 \geq \mu(\mathbb{H}P^2 \times \mathbb{R}^2) \geq 0.59\mu(\mathbb{S}^{10}) > 97.3 < \Lambda_{10,1}.$$

□

$n =$	3	4	5	6	7	8	9	10	11
$\sigma(M) \geq t_n =$	43.8	?	78.9	87.6	74.5	92.2	109.2	97.3	135.9
$\sigma(S^n) =$	43.8	61.5	78.9	96.2	113.5	130.7	147.8	165.0	182.1

TABLE 2. Lower estimates for the smooth Yamabe invariant of 2-connected manifolds with vanishing index. Values of $\sigma(S^n)$, rounded down, for comparison

In the case that $\alpha(M) \neq 0$ for 2-connected M it was shown in [15, Theorem 1] that $\sigma(M) = 0$.
In dimensions $n \leq 6$, $n \neq 4$, there are only a few 2-connected compact manifolds, namely S^3, S^5, S^6, and connected sums of $S^3 \times S^3$, all with their standard smooth structures. The conformal Yamabe constant for the product metric on $S^3 \times S^3$,

$$\mu(S^3 \times S^3, \rho^3 + \rho^3) = 12(2\pi^2)^{2/3} = 87.64646...,$$

follows from Obata's theorem [14, Proposition 6.2]. Using Theorem C or more precisely the third conclusion in the following unnumbered corollary of [8] we find

$$\sigma(S^3 \times S^3) > 12(2\pi^2)^{2/3} = 87.64646...$$

In all dimensions $\neq 4$ we thus obtain lower bounds for the smooth Yamabe invariant. In dimensions $n = 7$, $n = 8$, and $n \geq 11$ an explicit lower bound for the smooth Yamabe invariant of 2-connected compact manifolds with vanishing index was obtained in Corollaries 6.6, 6.7 and Proposition 6.9 of [2, Corollary 6.6]. Summarizing we have the following proposition.

Proposition 5.7. *Let M is a 2-connected compact manifold of dimension $n \neq 4$. If $\alpha(M) \neq 0$, then $\sigma(M) = 0$. If $\alpha(M) = 0$, then $\sigma(M) \geq t_n$, where t_n is an explicit positive number only depending on n.*

Some values of t_n are collected in Table 2.
The situation for $n = 4$ is still unclear as it is unknown whether exotic 4-spheres, i.e. manifolds homeomorphic but not diffeomorphic to S^4, do exist. The smooth Poincaré conjecture in dimension 4 claims that exotic 4-spheres do not exist. This would imply that S^4 is the only 2-connected 4-manifold and thus $t_4 = \sigma(S^4)$.

APPENDIX A. OPTIMAL VALUES OF λ AND τ

We now optimize λ and τ for the inequality (8). We define the convex polygon P_c of admissible pairs (λ, τ) as

$$P_c := \{(\lambda, \tau) \mid \text{satisfying (6), (7), } \lambda \geq 0,\ \tau \geq 0\}.$$

For $\lambda = 1$, $\tau = 0$, one has $\lambda c^2 s_1 + \tau s_0 < s_c$ so $(1, 0)$ is a corner of P_c. Similarly one sees that $(0, 1)$ is never a corner of P_c unless $c = 0$. Because of $c^2 s_1/s_0 < 1$, the equations $\lambda + \tau = 1$ and $\lambda c^2 s_1 + \tau s_0 = s_c$ have a common solution (λ_0, τ_0) with $\lambda_0 \in (0, 1)$ for $c \in (0, 1)$. From

$$\frac{c^{2w/n}\mu_1}{\mu_0} \geq c^{2w/n} > \frac{c^2 s_1}{s_0}$$

one easily sees that the optimal estimate is obtained in the point $(1, 0)$ for $c^{2w/n} \geq \mu_0/\mu_1$, and in the point (λ_0, τ_0) for $c^{2w/n} \leq \mu_0/\mu_1$.

Next we compute λ_0.

$$-\lambda_0 c^2 v(v-1) + \lambda_0 c^2 w(w-1) + (1-\lambda_0) w(w-1) \leq -c^2 v(v-1)$$

Factoring out, removing $w(w-1)$ on both sides, then dividing by $\lambda_0 c^2 w(w-1)$ one obtains the equivalent equation

$$-\frac{v(v-1)}{w(w-1)} + 1 - \frac{1}{c^2} \leq -\frac{1}{\lambda_0}\frac{v(v-1)}{w(w-1)},$$

which is further equivalent to

$$\left(1 - \frac{1}{c^2}\right) \leq \left(1 - \frac{1}{\lambda_0}\right)\frac{v(v-1)}{w(w-1)}.$$

This yields (10).

Appendix B. The constant $\Lambda_{6,3}$

All explicitly known positive lower bounds for $\Lambda_{n,k}$ are obtained in the following way: at first, we show that $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$ and then, we apply Theorem 4.1 or the estimates obtained in [3]. Recall that by definition $\Lambda_{n,k} = \min(\Lambda^{(1)}_{n,k}, \Lambda^{(2)}_{n,k})$. For $0 \leq k \leq n-2$ or $(n,k) \in \{(4,1),(5,2)\}$, the inequality $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$ is a direct consequence of the main result in [2]. For $(n,k) = (6,3)$, this result does not apply directly, but a modified version which will be presented in this appendix still allows to conclude $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$.

Proposition B.1. *We have $\Lambda^{(2)}_{6,3} \geq \Lambda^{(1)}_{6,3}$ and hence $\Lambda_{6,3} = \Lambda^{(1)}_{6,3}$.*

Proof. The main result in [2] implies that

$$\inf_{c\in[0,1)} \mu^{(2)}(\mathbb{M}_c) \geq \Lambda^{(1)}_{6,3}$$

and as a consequence, $\Lambda^{(2)}_{6,3} \geq \min(\Lambda^{(1)}_{6,3}, \mu^{(2)}(\mathbb{M}_1))$. We now estimate $\mu^{(2)}(\mathbb{M}_1)$. If we spell out the definition of $\mu^{(2)}(\mathbb{M}_1)$ recalled in Section 2.1, and using $a_6 = 5$, and $p_5 = 3$, we see that $\mu^{(2)}(\mathbb{M}_1)$ is the infimum of all $\mu \in \mathbb{R}$ for which there is a solution of

$$5\Delta^{G_1} u + s^{G_1} u = \mu u^2 \tag{14}$$

satisfying

- $u \not\equiv 0$,
- $\|u\|_{L^3(\mathbb{M}_1)} \leq 1$,
- $u \in L^\infty(\mathbb{M}_1)$,
- $\mu(u)\|u\|_{L^\infty(\mathbb{M}_1)} \geq \frac{5}{32}$.

We prove in [4] that there is a conformal diffeomorphism $\Theta : \mathbb{S}^6 \setminus \mathbb{S}^3 \to \mathbb{M}_1$ where $\mathbb{S}^3$ denotes a totally geodesic 3-sphere in the standard sphere $\mathbb{S}^6$. Let $f \in C^\infty(\mathbb{S}^6 \setminus \mathbb{S}^3)$, $f > 0$, be the conformal factor of Θ, i.e. $\Theta^* G_1 = f\rho^6$. We define $v := f\Theta^* u$. By conformal covariance of the Yamabe operator and since the scalar curvature of $\mathbb{S}^6$ is 30, we get from (14) that the function v is a solution of

$$5\Delta^{\rho^6} v + 30v = \mu v^2 \tag{15}$$

on $\mathbb{S}^6 \setminus \mathbb{S}^3$. Moreover, one checks that

$$\|v\|_{L^3(\mathbb{S}^6\setminus\mathbb{S}^3)} = \|u\|_{L^3(\mathbb{M}_1)}$$

and hence, $v \in L^3(\mathbb{S}^6)$ and

$$\|v\|_{L^3(\mathbb{S}^6)} \leq 1. \tag{16}$$

We now use a standard argument to show that the function v can be extended to a smooth solution of equation (15) on all S^6. In other words, we remove the singularity at $\mathbb{S}^3$. Let us choose a smooth function φ on $\mathbb{S}^6$. We are going to show that

$$\int_{S^6} v(L\varphi) - \mu v^2 \varphi \, dv = 0 \tag{17}$$

where, to simplify notations, we set $L := 5\Delta^{\rho_6} + 30$ and where $dv := dv^{\rho^6}$.
For all $a \geq 0$, let us denote by W_a the set of points of S^6 whose distance to the removed $\mathbb{S}^3$ is smaller than a. For this goal, consider for $\varepsilon \in (0, \frac{1}{2})$ a cut-off function η_ε such that

(i) $0 \leq \eta_\varepsilon \leq 1$;
(ii) $\eta_\varepsilon(\mathbb{S}^6 \setminus W_{2\varepsilon}) = \{0\}$;
(iii) $\eta_\varepsilon(W_\varepsilon) = \{1\}$;
(iv) $|\nabla \eta_\varepsilon| \leq 2/\varepsilon$.
(v) $|\nabla^2 \eta_\varepsilon| \leq c/\varepsilon^2$.

We then write, for $\varepsilon > 0$ small

$$\int_{S^6} v(L\varphi) \, dv = \int_{S^6} vL(\eta_\varepsilon \varphi + (1 - \eta_\varepsilon)\varphi) \, dv. \tag{18}$$

Since v satisfies Equation 15 and since the function $1 - \eta_\varepsilon$ is compactly supported in $\mathbb{S}^6 \setminus \mathbb{S}^3$, we have

$$\begin{aligned} \int_{S^6} vL((1 - \eta_\varepsilon)\varphi) \, dv &= \int_{\mathbb{S}^6} (Lv)(1 - \eta_\varepsilon)\varphi) \, dv \\ &= \int_{\mathbb{S}^6} \mu v^2 (1 - \eta_\varepsilon)\varphi) \, dv. \end{aligned}$$

Since $1 - \eta_\varepsilon$ is bounded and tends to 1 almost everywhere, Lebesgue's theorem implies

$$\lim_{\varepsilon \to 0} vL((1 - \eta_\varepsilon)\varphi) \, dv = \int_{\mathbb{S}^6} \mu v^2 \varphi \, dv. \tag{19}$$

Now, we use the fact that there exists some $C > 0$ independent of ε, but depending on φ, such that

$$L(\eta_\varepsilon \varphi) \leq C(\frac{\chi_\varepsilon}{\varepsilon^2} + \eta_\varepsilon)$$

where χ_ε is the characteristic function of the set $W_{2\varepsilon} \setminus W_\varepsilon$.

Then, using Hölder inequality and the fact that η_ε is compactly supported in $W_{2\varepsilon}$ and bounded by 1 on this set,

$$\begin{aligned}\int_{S^6} vL(\eta_\varepsilon\varphi)\,dv &\leq C\left(\frac{1}{\varepsilon^2}\int_{W_{2\varepsilon}} v\,dv + \int_{\mathbb{S}^6} v\eta_\varepsilon\,dv\right)\\ &\leq C\left(\frac{1}{\varepsilon^2}\left(\int_{W_{2\varepsilon}} v^3\,dv\right)^{1/3}\mathrm{vol}(W_{2\varepsilon})^{2/3} + \left(\int_{W_{2\varepsilon}} v^3\,dv\right)^{1/3}\mathrm{vol}(W_{2\varepsilon})^{2/3}\right)\\ &\leq C\frac{1}{\varepsilon^2}\left(\int_{W_{2\varepsilon}} v^3\,dv\right)^{1/3}\mathrm{vol}(W_{2\varepsilon})^{2/3}.\end{aligned}$$

Since $W_{2\varepsilon}$ is a 2ε-neighborhood of $\mathbb{S}^3$, $\mathrm{vol}(W_{2\varepsilon}) \leq C\varepsilon^3$. Moreover, since $v \in L^3(\mathbb{S}^6)$,

$$\lim_{\varepsilon\to 0}\int_{W_{2\varepsilon}} v^3\,dv = 0.$$

We then obtain that

$$\lim_{\varepsilon\to 0}\int_{S^6} vL(\eta_\varepsilon\varphi)\,dv = 0.$$

Together with (19) and (18), we obtain (17) which means that in the sense of distributions, equation (15) is satisfied on all of $\mathbb{S}^6$. By standard elliptic theory, v is C^2 (and even smooth outside its zero set). Using v as a test function in the Yamabe function of $\mathbb{S}^6$, we get from (15) and (16) that $\mu \geq \mu(\mathbb{S}^6) \geq \Lambda^{(1)}_{6,3}$, which ends the proof. □

Appendix C. The Wu manifold SU(3)/SO(3)

We equip SU(3) with the bi-invariant metric such that the matrix

$$\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \in \mathfrak{su}(3)$$

has length 1. Then $(\mathrm{SU}(3), \mathrm{SO}(3))$ is a symmetric pair, and the associated involution of $\mathfrak{su}(3)$ is complex conjugation. Let M be SU(3)/SO(3) equipped with the quotient metric g. The manifold M is orientable, but not spin. Complex conjugation of SU(3) induces an orientation reversing isometry of M. Thus $M \amalg M$ is (up to orientation-preserving diffeomorphisms) the oriented boundary of $M \times [0,1]$. It follows that $M \# M$ is an oriented boundary as well.
An elementary calculation on the Lie algebra level shows that g is an Einstein metric, $\mathrm{Ric}^g = 6g$. Obata's theorem [14, Proposition 6.2] then tells us that $\mu(M,g) = 30\mathrm{vol}(M,g)^{2/n}$. The volume $\mathrm{vol}(M,g)$ is calculated in [9], and we conclude the following Lemma.

Lemma C.1. *The conformal Yamabe constant of* SU(3)/SO(3) *is*

$$\mu(\mathrm{SU}(3)/\mathrm{SO}(3), g) = 30\cdot\left(\frac{\sqrt{3}}{8}\pi^3\right)^{2/5} = 64.252401...$$

Appendix D. Quaternionic projective spaces $\mathbb{H}P^n$

Let g_n be the metric on $\mathbb{H}P^n$ such that the Hopf map $\mathbb{S}^{4n+3} \to \mathbb{H}P^n$ is a Riemannian submersion. With O'Neill's formula one easily calculates that the scalar curvature of g_n is $s^{g_n} = 4n(4n+8)$, and the volume is $\mathrm{vol}(\mathbb{H}P^n, g_n) = \omega_{4n+3}/\omega_3$.
As a consequence

$$\tilde{g}_n := \frac{s^{g_n}}{s^{\rho^{4n}}} g_n = \frac{4n(4n+8)}{4n(4n-1)} g_n$$

is a metric whose scalar curvature is equal to $4n(4n-1) = s^{\rho^{4n}}$. Its volume is

$$V_{4n} := \mathrm{vol}(\mathbb{H}P^n, \tilde{g}_n) = \left(\frac{4n+8}{4n-1}\right)^{2n} \frac{\omega_{4n+3}}{\omega_3}.$$

In the special case $n = 2$ this yields $V_8 = 2^{13}\pi^4/(7^4 \cdot 5 \cdot 3)$ where we used $\omega_{11} = \pi^6/60$ and $\omega_3 = 2\pi^2$. Using $\omega_8 = 32\pi^4/(7 \cdot 5 \cdot 3)$ we obtain $V_8/\omega_8 = 2^8/7^3 = 0.74635569\ldots$. These numbers play a crucial role for the lower bounds of $\mu(\mathbb{H}P^2 \times \mathbb{R})$ and $\mu(\mathbb{H}P^2 \times \mathbb{R}^2)$.

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